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The Structure of Cities

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Abstract. We give an account of recent work, where the spatial structure of cities is analysed as an equilibrium problem between the various uses of land. The mathematical theory of optimal transportation plays a crucial role in the proof.

1. The Problem

Cities have a structure: there are business centers and residential areas. There are also mixed areas, where businesses and residences can be found together. Can this structure be explained by economic arguments? This problem was first addressed quite recently by Lucas and Rossi-Hansberg [4]. They found an equilibrium configuration in the radially symmetric case, that is, when the city has circular shape and the three types of areas are assumed to be annuli. In recent work, Carlier and Ekeland [2] have been able to find equilibrium configurations in general situations. The purpose of this talk is to report on this work.

2. The Model

The model consists of workers, firms and land owners who have to fit inside a given open bounded subset of R^2, Ω (the city). A single good is produced within the city. It serves both as consumption good and numéraire. No equilibrium is sought on the goods market: any amount that is not consumed within the city is exported. The equilibrium will be sought in the land market, where businesses and residences compete, and this is what will determine the structure of the city.

All inhabitants of the city are identical, Their preferences are given by a utility function $(c,S)\mapsto U(c,S)$ defined over the quantity c consumed, and the area S of land occupied. We assume that U is strictly concave, increasing in both arguments and continuously differentiable.

All firms in the model are identical. They have the same technology with constant returns to scale. Their production function per unit of business land at a location x is given by a function $(z,n)\mapsto f(z,n)$ where $z=z(x)$ is the productivity at x and $n=n(x)$ is the number of workers per unit of business land. It is assumed that f is continuously differentiable, strictly increasing with respect to both arguments and strictly concave with respect to n.

Land owners who are offered a rent per surface unit Π for business use and a rent Q for residential use determine a fraction $\theta(\Pi, Q)$ of land used for production so that $1-\theta(\Pi, Q)$ is the fraction of land for residential use. As an example, we set:

$$\theta=1 \quad \text{if } \Pi > Q$$
$$\theta=0 \quad \text{if } \Pi < Q$$

so that landlords simply rent the land to the highest bidder. If $\Pi=Q$, they are indifferent, and the values of $\theta(x)$ in that region will be determined by the equilibrium conditions.

The main features of the model, which are already to be found in the paper of Lucas and Rossi-Hansberg, are transportation costs and externalities of production. However, our view of transportation costs is quite different from theirs, and this is what enables us to go beyond the symmetric case.

There is a transportation cost for workers: if they live at x and work at y, they pay $c(x-y)$. The function c: $R^2 \to R$ is assumed to be strictly convex and C^1.

There is a production externality: having many workers together increases the productivity of each. The productivity is given by

$$z(x) := \chi\left(\int_\Omega \rho(x,y)\nu(y)\,\mathrm{d}y\right) \tag{1}$$

where $\nu(x)\mathrm{d}x$ is the number of workers per unit of land, ρ some given kernel approximating the Dirac mass, and χ: $R \to R$ is increasing

From then on, everyone behaves rationnally: workers maximize their utility, firms maximize their profits, landlords maximize their rent.

3. Equilibrium

We want to know if there is a situation where no one has any incentive to move: this will be the structure of the city. In this situation, there must be no way for workers to increase their utility or to firms to increase their profits. This implies that all inhabitants must have the same utility $\bar{u}$ (otherwise, since they are all identical, they would try to emulate those who do better) and all firms must make the same profit (which we take to be 0, meaning that all revenues from the business goes as rent to the landlords).

An inhabitant who lives at x and brings home φ solves the problem:

$$\operatorname*{Max}_{(S,c)}\{U(S,c) \mid QS + c \geqslant \varphi\}$$

yielding utility $\bar{u}$. From this we can compute his consumption level $c(\varphi)$, the space he occupies $S(\varphi)$, the rent he is willing to pay $Q(\varphi)$, and we can also compute $m(x)$, the number of dwellers per unit of residential land at x.

If a firm is established at y, where productivity is z, and pays wages ψ, it solves the problem

$$\operatorname*{Max}_{\nu}\{f(z,n)-n\psi \mid n\geqslant 0\}$$

from which we can determine $n(z,\psi)$ the number of workers per unit of business land at y and the rent $\Pi(z,\psi)$ the firm can pay at y.

Land owners determine a fraction of land $\theta(\Pi(z,\psi),Q(\varphi))$ devoted to business use. Let us denote it by $\theta(z,\psi,\varphi)$. We then get densities of workers and residents ν and μ from the formulas;

$$\nu=\theta n$$

$$\mu=(1-\theta)m$$

An *equilibriu* is a pair $(\phi(x),\psi(y))$ such that:

$$z(x)=\chi\left(\int_{\Omega}\rho(x,y)\nu(z(y),\psi(y))\right)\mathrm{d}y$$

densities of residents $\mu(x)$ and densities of jobs $\nu(x)$ are determined by land owners through the rents $Q(\varphi(x))$ and $\Pi(z(y),\psi(y))$, there is conservation of mass:

$$\int\mu(x)\mathrm{d}x=\int\nu(y)\mathrm{d}y$$

and free mobility of labor.

The conservation of mass (the number of residents equals the number of workers) is a feature of our model which does not hold in the model of Lucas and Rossi-Hansberg, and it will turn out to be mathematically very useful. By free mobility of labor, we mean that workers have no incentive to change the place they work at; the consequences of this assumption will be spelt out in the next section.

4. Free Mobility of Labor

Let $\varphi(x)$ be the available revenue at location x, and $\psi(y)$ is the wage paid at location y. Workers look for the best location to work at:

$$\varphi(x)=\sup_{y\in\Omega}\{\psi(y)-c(x,y)\} \tag{2}$$

and firms try to hire workers away from other firms:

$$\psi(y)=\inf_{x\in\Omega}\{\varphi(x)+c(x,y)\} \tag{3}$$

The solution to (w) should be a map $y=s(x)$ which tells us were the people who live at x work. The solution to (f) should be a map $x=t(y)$ which tells us where the people who work at y live. We should have $s=t^{-1}$ and $t=s^{-1}$. In addition, the maps s and t should transform the density of residents μ into the density of workers ν

$$\nu=s(\mu), \quad \mu=t(\nu)$$

It is surprising that maps $\varphi(x)$ and $\psi(y)$ should exist which satisfy the above conditions. It is true, however, and it is a consequence of the theory of optimal transportation, which we now describe.

Given (μ,ν) two continuous positive measures on Ω with same total mass $\int\mu=\int\nu$ and a cost function of the form $c(x,y)=C(x-y)$ with C strictly convex and differentiable, consider the following pair of problems, denoted respectively by $(\mathrm{Primal})_{\mu,\nu}$ and $(\mathrm{Dual})_{\mu,\nu}$:

$$\begin{aligned}&\inf_s\int_\Omega C(x-s(x))\mu(x)\,\mathrm{d}x\\ &s\colon \Omega\to\Omega, \quad s(\mu)=\nu\end{aligned} \tag{4}$$

and

$$\begin{aligned}&\sup_{\varphi,\psi}\left\{\int_\Omega\psi\nu\,\mathrm{d}y-\int_\Omega\varphi\mu\,\mathrm{d}x\right\}\\ &\psi(y)-\varphi(x)\leqslant C(x-y),\forall(x,y)\in\Omega^2\end{aligned} \tag{5}$$

THEOREM 4.1. *The supremum in* $(\mathrm{Dual})_{\mu,\nu}$ *is attained by a pair* (ψ,φ) *of conjugate functions, and the duality relation*

$$\inf(\mathrm{Primal})_{\mu,\nu}=\sup(\mathrm{Dual})_{\mu,\nu}$$

holds. In addition, there exists a unique map $s\colon\Omega\to\Omega$ *such that*

$$\varphi(x)=\psi(s(x))-C(x-s(x))$$

the map s sends μ on ν, and s is the unique solution of $(\mathrm{Primal})_{\mu,\nu}$. *Finally, s is invertible and $t=s^{-1}$ satisfies*:

$$\psi(y)=\varphi(t(y))+C(t(y)-y).$$

We refer to [3, 5] and [1] for a proof of this result.

5. Existence of an Equilibrium

Set:

$$\Delta := \left\{(\mu,\nu)\in C^0_{++}(\overline{\Omega})^2, \int_\Omega \mu\,\mathrm{d}x = \int_\Omega \nu\,\mathrm{d}y\right\}$$

and define an operator T: $\Delta \to \Delta$ as follows.

Start from $(\mu,\nu)\in\Delta$ a pair of densities.

Step 1. Define the productivity function z by:

$$z = \chi\left(\int_\Omega \rho(x,y)\nu(y)\,\mathrm{d}y\right)$$

Step 2. Let (ψ,φ) be the only solution of problem $(\mathrm{Dual}_{\mu,\nu})$ such that $\min\psi = 1$. Notice that ψ,φ are continuous functions and that $\varphi \geqslant \psi \geqslant 1$.

Step 3. Define new densities by:

$$S(\varphi)\tilde{\mu}(z,\psi,\varphi) := (1-\theta(z,\psi,\varphi)) \tag{6}$$

$$\tilde{\nu}(z,\psi,\varphi) := \theta(z,\psi,\varphi)n(z,\psi) \tag{7}$$

Step 4. Find the only constant λ such that:

$$\int_\Omega \tilde{\mu}(Z_\nu(x),\psi(x)+\lambda,\varphi(x)+\lambda)\,\mathrm{d}x$$
$$= \int_\Omega \tilde{\nu}(Z_\nu(x),\psi(x)+\lambda,\varphi(x)+\lambda)\,\mathrm{d}x$$

It turns out that $T_2(\psi,\varphi)$ is a pair of Lipschitz-continuous strictly positive functions.

Step 5.

$$\mu'(x) := \tilde{\mu}(Z_\nu(x),\psi(x)+\lambda,\varphi(x)+\lambda)$$
$$\nu' := \tilde{\nu}(Z_\nu(x),\psi(x)+\lambda,\varphi(x)+\lambda)$$

It is immediate to check that by construction $(\mu',\nu') = T(\mu,\nu)\in\Delta$.

An equilibrium is a fixed point of T. By Ascoli's theorem, the map T is easily seen to be compact. Existence is proved by Schauder's fixed-point theorem.

References

1. Carlier, G. Duality and existence for a class of mass transportation problems and economic applications, *Adv. in Math. Econ.* to appear
2. Carlier, G. and Ekeland, I. (2002), *Optimal Transportation and the Structure of Cities*, Working Paper.
3. Gangbo, W. (1994), An elementary proof of the polar factorization of vector-valued functions, *Arch. Rational Mech. Anal.*, 128, 381–399.
4. Lucas, R.E. Jr. and Rossi-Hansberg, E. (2002), On the internal structure of cities, *Econometrica*, 70, 1445–1476.
5. Rachev, S.T. and Rüschendorf, L. (1998), *Mass Transportation Problems. Vol. I: Theory, Vol. II: Applications*, Springer-Verlag,